IUCAA GR Refresher Course Tutorials by BM

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Tutorial:3 (26-06-2025)

Problem 1: Geodesics on Curved Surfaces

Problem (a): Geodesics on a Cylinder

Consider a cylinder of fixed radius R embedded in $\mathbb{R}^3$, parametrized by coordinates (ϕ, z) where $\phi \in [0, 2\pi)$ and $z \in \mathbb{R}$. The induced metric on the cylindrical surface is:

$$ds^2 = R^2 d\phi^2 + dz^2$$

Tasks:

1. Derive the geodesic equations from the given metric.
2. Show that the general geodesic is a **helix**, i.e., a curve of constant pitch winding around the cylinder.
3. Identify the special cases under which the geodesic becomes either:
   - A **circle** (constant z), or
   - A **straight line** along the axis (constant ϕ).

Problem (b): Geodesic Equations on a 2-Sphere

Consider a 2-dimensional sphere of radius R with spherical coordinates (θ, ϕ), where $\theta \in (0, \pi)$ and $\phi \in [0, 2\pi)$. The line element is:

$$ds^2 = R^2 \left(d\theta^2 + \sin^2\theta \, d\phi^2\right)$$

Tasks:

1. Derive the geodesic equations from the above metric using the Euler–Lagrange formalism or Christoffel symbols.
2. Show that the equations reduce to a second-order ODE for $\theta(\phi)$.
3. Interpret the result as the equation of an **orbit** on the surface of the sphere.
4. Identify the geodesics as **great circles**, and verify that $\theta = \pi/2$ (equatorial path) is one such solution.

Problem:2 Derivation of the Geodesic Deviation Equation

In General Relativity, free-falling test particles follow geodesics in curved spacetime. To understand how nearby geodesics diverge or converge due to spacetime curvature, we consider a **one-parameter family of geodesics** and study the behavior of the **deviation vector** between them.
Derive the **geodesic deviation equation**:

$$\frac{D^2\xi^\mu}{D\tau^2} = R^\mu{}_{\nu\rho\sigma} u^\nu u^\rho \xi^\sigma$$

where $\frac{D}{D\tau}$ is the covariant derivative along the geodesic and $R^\mu{}_{\nu\rho\sigma}$ is the Riemann curvature tensor.

Problem:3 Planarity of Geodesics in Schwarzschild Spacetime

The Schwarzschild metric describes the spacetime geometry outside a static, spherically symmetric mass and is given by:

$$ds^2 = -\left(1 - \frac{2GM}{r}\right) dt^2 + \left(1 - \frac{2GM}{r}\right)^{-1} dr^2 + r^2 \left(d\theta^2 + \sin^2\theta\, d\phi^2\right)$$

Show that the **geodesic motion of a free-falling test particle in Schwarzschild spacetime is always confined to a plane**. That is, demonstrate that without loss of generality, one can choose $\theta = \frac{\pi}{2}$ for the entire trajectory.

Hints:

- Use the symmetry of the Schwarzschild metric under rotations.
- Consider the conservation of angular momentum.
- Analyze the geodesic equation for θ and argue that initial conditions $\theta = \frac{\pi}{2}$, $\frac{d\theta}{d\tau} = 0$ imply the motion remains in the equatorial plane.

This question highlights how spacetime symmetries (spherical symmetry in this case) reduce the complexity of geodesic equations and reflects a deep physical insight: motion in central force fields is planar.